

Diagrams for Heat Kernel Expansions.

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Abstract

A diagrammatic heat kernel expansion technique is presented. The method is especially well suited to the small derivative expansion of the heat kernel, but it can also be used to reproduce the results obtained by the approach known as covariant perturbation theory. The new technique gives an expansion for the heat kernel at coincident points. It can also be used to obtain the derivative of the heat kernel, and this is useful for evaluating the expectation values of the stress-energy tensor.

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I. INTRODUCTION

The heat kernel was first introduced into quantum field theory to help evaluate the polarization of the quantum vacuum caused by an external field [1]. The heat kernel method invented by Schwinger was developed by DeWitt into one of the key techniques for evaluating the one loop effective action in the background field approach [2,3].

In many applications, the heat kernel is replaced by its asymptotic series in powers of the proper time [2,4,5]. The coefficients in this series relate directly to physical quantities, with the first coefficient corresponding to the Weyl formula for the spectral density and further coefficients being relevant to the renormalisation group and to anomalies [6]. The coefficients depend on external potentials, the field strengths of external gauge fields and the Riemannian curvature of the underlying space. The first six coefficients have been calculated by a wide variety of methods [2,7–10].

In practice, the proper-time expansion of the heat kernel is usually inadequate for obtaining a good approximation to the effective action but an expansion in powers of covariant derivatives of the background fields may suffice. The terms in such a series can be organised in various ways depending on which quantities are to be regarded as being small. If the field strengths and derivatives of the external potentials are small then an expansion in powers of derivatives will be appropriate. This expansion is essentially a low energy expansion.

Schwinger's original calculation of the vacuum polarization in a constant external field corresponds to a slightly different type of expansion where derivatives of the field strength can be neglected but the field strength itself may be large. This type of expansion leads directly to the Euler-Heisenberg effective action.

A complimentary situation would be where the field strengths were small compared to their derivatives. The expansion should then be arranged in increasing powers of the curvatures with infinite numbers of derivatives resummed at each stage. This large derivative or high energy expansion is possible and it is used in an approach known as covariant perturbation theory [11,12].

In this paper we present a new diagrammatic expansion technique, which is basically a small derivative expansion of the heat kernel, but which can also be used to reproduce the results obtained by covariant perturbation theory. The new technique gives an expansion for the heat kernel at coincident points. It can also be used to obtain local derivatives of the heat kernel, and this is useful for evaluating the expectation values of the stress-energy tensor [13].

There are close affinities between our method and the method invented by Barvinski and Vilkoviski [12] and developed by Barvinski, Osborn and Gusev [14]. In all of these methods, the diagrams are essentially the Feynman diagrams for a non-relativistic particle interacting with a background potential. However, there are important differences between the method presented here and previous methods for expanding the heat kernel. Firstly, we have organised our expansion into powers of derivatives of the potential, rather than the proper time. Our rules give the coincident heat kernel, whereas results in [12] are given only for the integrated heat kernel. Furthermore, our rules apply in curved space, whereas results in [14] are for flat space, although a curved space generalisation was sketched out.

We will present various new results including new terms in the derivative expansion in flat space and for constant background gauge field strengths. We also give new terms in the

normal coordinate expansion of the metric that can be used elsewhere.

II. THE BASIC METHOD

Consider a Riemannian manifold (\mathcal{M}, g) , and a second order operator Δ acting on a set of fields $\phi(x)$. We are interested in the heat kernel $G(x, x', t)$, defined by

$$\Delta G + \dot{G} = 0, \quad G(x, x', 0) = \delta(x, x') \quad (1)$$

where a dot denotes derivatives with respect to the proper time t and $\delta(x, x')$ is the covariant delta function.

Throughout this section the operator will be taken to have a specific form

$$-g^{\mu\nu} \partial_\mu \partial_\nu + \Gamma^\mu \partial_\mu + W \quad (2)$$

where ∂_μ is the partial derivative with respect to x^μ . The aim of this section is to obtain a series expansion for the heat kernel in powers of partial derivatives of $g^{\mu\nu}$, Γ^μ and W .

Our method begins in the way introduced by Parker [17] and follows some steps of reference [15]. We define

$$K(k, x', t) = \int d\mu(x) e^{-ik(x-x')} G(x, x', t) \quad (3)$$

Taking the Fourier transform of equation (1) and integrating by parts gives,

$$\dot{K}(k, x', t) = - \int d\mu(x) (k^2 + W - Z) G(x, x', t) e^{-ik(x-x')} \quad (4)$$

$$Z(k, x, x') = (\partial_\mu - ik_\mu)(\partial_\nu - ik_\nu) h^{\mu\nu} + (\partial_\mu - ik_\mu) \Gamma^\mu - W(x) + W(x') \quad (5)$$

and $h^{\mu\nu} = g^{\mu\nu} - \delta^{\mu\nu}$. Expanding Z in a Taylor series about $x = x'$ gives

$$\dot{K}(k, x', t) = - \left(k^2 + W - Z(k, D) \right) K(k, x', t) \quad (6)$$

where

$$Z(k, D) = \sum_{r=0}^{\infty} \frac{1}{r!} Z_{,\mu_1 \dots \mu_r} D^{\mu_1} \dots D^{\mu_r} \quad (7)$$

and $D^\mu = i\partial/\partial k_\mu$.

Equation (6) has a solution in terms of a time-ordered exponential,

$$K = e^{-(k^2+W)t} T \exp \left(\int_0^t e^{(k^2+W)t'} Z(k, D) e^{-(k^2+W)t'} dt' \right) \quad (8)$$

Redistributing some of the terms gives a more convenient form

$$K = e^{-Wt} e^{-k^2 t/2} T \exp \left(\int_0^t e^{Wt'} Z(\tfrac{1}{2} i \delta', \delta) e^{-Wt'} dt' \right) e^{-k^2 t/2} \quad (9)$$

where

$$\delta^\mu(t') = D^\mu - 2ik^\mu t' + ik^\mu t \quad (10)$$

and $\delta' = -2ik$. The function Z can be simplified by using the identity

$$[F(\delta), \delta'_\mu] = 2F(\delta)_{,\mu}, \quad (11)$$

which is valid for any function $F(\delta)$, to remove some of the derivatives. After this simplification,

$$Z = \frac{1}{4}h^{\mu\nu}(\delta)\delta'_\mu\delta'_\nu + \frac{1}{2}\Gamma^\mu(\delta)\delta'_\mu - W(\delta) \quad (12)$$

The coincidence limit of the heat kernel is obtained by integration

$$G(x, x, t) = \int d\mu(k) K(k, x, t) \quad (13)$$

where $d\mu(k) = d^n k / (2\pi)^n$ in n dimensions. Coincidence limits of derivatives of the heat kernel can be obtained in a similar way, for example

$$[\partial_\mu G] = - \int d\mu(k) \frac{1}{2} \delta'_\mu K(k, x, t) \quad (14)$$

$$[\partial_{\mu'} G] = \int d\mu(k) K(k, x, t) \frac{1}{2} \delta'_\mu \quad (15)$$

where $[\dots]$ denotes evaluation at $x' = x$. (The operators in equation (9) act on the δ' in equation (15)). Extra derivatives introduce extra factors of δ' .

We shall define

$$\langle \dots \rangle = \frac{1}{K_0(t)} \int d\mu(k) e^{-k^2 t/2} \dots e^{-k^2 t/2} \quad (16)$$

where the normalisation factor

$$K_0(t) = (4\pi t)^{-d/2} \quad (17)$$

for d dimensions. The coincident limit of the heat kernel is therefore

$$G(x, x, t) = K_0(t) e^{-Wt} \left\langle T \exp \left(\int_0^t e^{Wt'} Z(\frac{1}{2}i\delta', \delta) e^{-Wt'} dt' \right) \right\rangle. \quad (18)$$

Equation (18) is the basic equation for obtaining the derivative expansion of the heat kernel. It yields a convenient diagrammatic expansion by the conventional route using Wicks theorem. It is similar to an equation (for a more restricted operator) used in reference [14].

We can expand the exponential and introduce creation and annihilation operators $c_\pm(k, D)$ that satisfy

$$c_-^\mu e^{-k^2 t/2} = 0, \quad [c_-^\mu, c_+^\nu] = \delta^{\mu\nu} \quad (19)$$

The operators δ and δ' can then be replaced by c_\pm ,

$$\delta^\mu(t') = \sqrt{\frac{2}{t}} \{ (t-t')c_-^\mu + t'c_+^\mu \} \quad (20)$$

By Wick's theorem, the time-ordered product of operators can be replaced by products of propagators. There are three different propagators corresponding to the distinct ways of combining the operators,

$$\langle T \delta^\mu(t_i) \delta^\nu(t_j) \rangle = \delta^{\mu\nu} D(t_i, t_j) \quad (21)$$

$$\langle T \delta'_\mu(t_i) \delta^\nu(t_j) \rangle = \delta_\mu^\nu \vec{D}(t_i, t_j) \quad (22)$$

$$\langle T \delta'_\mu(t_i) \delta'_\nu(t_j) \rangle = \delta_{\mu\nu} \overleftrightarrow{D}(t_i, t_j) \quad (23)$$

These can be evaluated using the creation and annihilation operators,

$$D(t_i, t_j) = 2\min(t_i, t_j) - 2t^{-1}t_it_j \quad (24)$$

$$\vec{D}(t_i, t_j) = 2\theta(t_j - t_i) - 2t^{-1}t_j \quad (25)$$

$$\overleftrightarrow{D}(t_i, t_j) = -2t^{-1} \quad (26)$$

where $\theta(t) = 1$ for $t \geq 0$ and zero otherwise.

After applying Wick's theorem, a typical term in the expansion of equation (18) will take the form

$$K_0(t)e^{-Xt} \int dt_1 \dots dt_n T \left(e^{Xt_1} Z_1 e^{-Xt_1} \dots e^{Xt_n} Z_n e^{-Xt_n} \right) g(t_1 \dots t_n) \quad (27)$$

where $g(t_1 \dots t_n)$ is a combination of propagators and index contractions. The factors Z_i arise from the Taylor series expansion of Z (see (12)). When defining these factors it is convenient to multiply each term in the Taylor series by $p!r!$, where p is the order in δ' and r the order in δ , as this simplifies the combinatoric factors in the final expansion.

The terms in the heat kernel expansion can be represented by diagrams, with lines representing propagators and vertices representing the terms Z_i . There are three types of vertex, corresponding to the three terms in equation (12). Each term is divided by a numerical factor corresponding to the order of the symmetry group of the diagram in the usual way.

III. COVARIANT EXPANSIONS

Second order operators are often written in the form

$$\Delta = -D^2 + X \quad (28)$$

where $D = \nabla + iA$ is a gauge covariant derivative, with gauge connection A and Levi-Civita connection Γ . The heat kernel expansion can then be expressed in terms of curvatures and covariant derivatives. We will use the notation $O(D^n)$ to describe the size of any term in the heat kernel expansion which has n covariant derivatives and any number of factors of X . The field strengths will be formally $O(D)$.

The partial derivative expansion can be obtained from the method just described. With this particular operator the vertices are generated by a function \bar{Z} , where (see equation (12)),

$$\begin{aligned}\bar{Z} = & \frac{1}{4}h^{\mu\nu}\delta'_\mu\delta'_\nu + \frac{1}{2}\Gamma^\mu\delta'_\mu - \frac{1}{2}i\delta^{\mu\nu}\{\delta'_\mu, A_\nu\} - X \\ & - \frac{1}{2}i\{h^{\mu\nu}, \delta'_\mu\}A_\nu - i\Gamma^\mu A_\mu - g^{\mu\nu}A_\mu A_\nu\end{aligned}\quad (29)$$

We have made use of the identity (11) in this expression. The heat kernel expansion is generated by expanding equation (18). The anticommutators $\{, \}$ only affect contractions at equal times, and this is taken into account in the diagrammatic rules given below.

In principle, each of the terms generates a series of vertices when expanded in powers of δ . However, most of the terms in \bar{Z} can be eliminated if the propagator \vec{D} attached to a vertex formed from the gauge field A is replaced by the propagator

$$\vec{D}_m(t_i, t_j) = 2\delta(t_i - t_j) - 2t^{-1} \quad (30)$$

The vertices are generated by

$$Z = \frac{1}{4}h^{\mu\nu}\delta'_\mu\delta'_\nu + \frac{1}{2}\Gamma^\mu\delta'_\mu - \frac{1}{2}i\delta^{\mu\nu}\{\delta'_\mu, A_\nu\} - X \quad (31)$$

This is particularly useful because the new propagator is the derivative of \vec{D} . The details are given in appendix A.

Each of the vertices in the diagrammatic expansion corresponds to a multiple partial derivative of Z . It is convenient to define vertex functions by

$$D_\xi^r Z = \xi^{\mu_1} \dots \xi^{\mu_r} Z_{,\mu_1 \dots \mu_r} \quad (32)$$

In a system of normal coordinates the vertex functions can also be expressed in terms of curvatures and field strengths, as described in appendix B.

The rules for evaluating the heat kernel $G(x, x, t)$ consist of drawing all possible diagrams with lines from figure 1 and vertices from figure 2. The i 'th vertex is associated with its own time variable t_i , where $0 \leq t_i \leq t$. The expression corresponding to the diagram has

(1) For each line, stretching from a vertex i to vertex j , one of the propagators

$$D(t_i, t_j) = 2\min(t_i, t_j) - 2t^{-1}t_it_j \quad (33)$$

$$\vec{D}(t_i, t_j) = 2\theta(t_j - t_i) - 2t^{-1}t_j \quad (34)$$

$$\vec{D}(t_i, t_j) = \beta\delta(t_i - t_j) - 2t^{-1} \quad (35)$$

depending on the number of arrows. In these expressions, $\theta(0) = 1$ and $\beta = 0$ if only g and ω vertices are attached, whilst $\theta(0) = \frac{1}{2}$ and $\beta = 2$ if there is an A vertex.

(2) For each vertex, the coefficient of the appropriate vertex function

$$\frac{1}{2}e^{Xt_i}D_\xi^r h^{\mu\nu}e^{-Xt_i} \quad (36)$$

$$\frac{1}{2}e^{Xt_i}D_\xi^r \Gamma^\mu e^{-Xt_i} \quad (37)$$

$$-ie^{Xt_i}D_\xi^r A_\mu e^{-Xt_i} \quad (38)$$

$$-e^{Xt_i}D_\xi^r X e^{-Xt_i} \quad (39)$$

These vertex functions can be expressed in terms of covariant tensors by using tables I to III. The components of these tensors are contracted according to the arrangement of the lines.

The differentiated heat kernel at two coincident points is represented by diagrams with an external line for each derivative of the heat kernel. The free ends of the x and x' derivative lines are associated with the time variables t and 0 respectively. (This respects the ordering of the terms in equation (15)). These contribute

(3) For each external line, the appropriate propagator from rule (1) multiplied by $\frac{1}{2}$ for an x' and $-\frac{1}{2}$ for an x derivative.

The resulting expression after applying rules (1)-(3) has to be time ordered and integrated with respect to the time variables. It is then multiplied by $K_0 \exp(-Xt)$ (see equation (17)) and divided by the order of the symmetry group of the diagram.

The expansion obtained from these diagrams is a derivative expansion where a diagram with n internal lines and m arrows produces a term of $O(D^{2n-m})$.

IV. ABELIAN THEORIES IN FLAT SPACE

The evaluation of the heat kernel simplifies considerably for abelian gauge theories in flat space. In some situations it is even possible to obtain exact results for the heat kernel, for example with a constant field strength F . In this section we will give the leading terms in a small derivative expansion of the heat kernel and show how resummations can be performed to generalise Schwinger's result for constant fields. We also show how resummations can be used to obtain an expansion for the heat kernel in powers of the curvatures, which can be regarded as a large derivative or high energy expansion [12].

The most significant simplification for abelian theories is that only the connected diagrams need be evaluated, since the results from the connected diagrams can be exponentiated to reproduce the results from the complete set of diagrams. This follows from the same simple counting argument that applies to the usual Feynman graph expansion.

Other important simplifications include the fact that the vertex functions can be given explicitly,

$$D_\xi^{n-1} A_\mu = \frac{n-1}{n} F_{\nu_1 \mu, \nu_2 \dots \nu_{n-1}} \xi^{\nu_1} \dots \xi^{\nu_{n-1}} \quad (40)$$

$$D_\xi^n X = X_{, \nu_1 \dots \nu_n} \xi^{\nu_1} \dots \xi^{\nu_n} \quad (41)$$

We can also reduce the number of diagrams by integration by parts. This is possible because of the relationships

$$\vec{D}(t_i, t_j) = \frac{\partial}{\partial t_i} D(t_i, t_j) \quad (42)$$

$$\overleftrightarrow{D}(t_i, t_j) = \frac{\partial}{\partial t_j} \vec{D}(t_i, t_j) \quad (43)$$

that enable us to eliminate diagrams with \overleftrightarrow{D} .

The diagrams for some terms in the small derivative expansion with $F = 0$ are shown in figure 3. We write the small derivative expansion of the heat kernel in the form

$$G(x, x, t) = K_0(t) \exp \left(\sum_{n=0}^{\infty} W_n \right) \quad (44)$$

Results up to $O(D^6)$ can be obtained by drawing all diagrams with up to three lines and using table IV to evaluate the time integrals,

$$W_0 = -Xt \quad (45)$$

$$W_2 = -\frac{1}{6}(\partial^2 X)t^2 + \frac{1}{12}(\partial X)^2 t^3 \quad (46)$$

$$W_4 = -\frac{1}{60}(\partial^4 X)t^3 + \frac{1}{90}(\partial_\mu \partial_\nu X)(\partial^\mu \partial^\nu X)t^4 \\ + \frac{1}{30}(\partial^\mu X \partial_\mu \partial^2 X)t^4 - \frac{1}{30}(\partial^\mu X)(\partial^\nu X)(\partial_\mu \partial_\nu X)t^5 \quad (47)$$

$$W_6 = -\frac{1}{840}(\partial^6 X)t^4 + \frac{1}{480}(\partial_\mu \partial^4 X)(\partial^\mu X)t^5 \\ + \frac{1}{840}(\partial_\mu \partial_\nu \partial_\rho X)(\partial^\mu \partial^\nu \partial^\rho X)t^5 + \frac{1}{210}(\partial_\mu \partial_\nu \partial^2 X)(\partial^\mu \partial^\nu X)t^5 \\ + \frac{17}{5040}(\partial_\mu \partial^2 X)(\partial^\mu \partial^2 X)t^6 - \frac{17}{2520}(\partial_\mu X)(\partial_\nu \partial^2 X)(\partial^\mu \partial^\nu X)t^6 \\ - \frac{1}{480}(\partial_\mu X)(\partial_\nu X)(\partial^\mu \partial^\nu \partial^2 X)t^6 - \frac{1}{210}(\partial_\mu X)(\partial_\nu \partial_\rho X)(\partial^\mu \partial^\nu \partial^\rho X)t^6 \\ - \frac{8}{2835}(\partial^\mu \partial_\nu X)(\partial^\nu \partial_\rho X)(\partial^\rho \partial_\mu X)t^6 + \frac{1}{840}(\partial_\mu X)(\partial_\nu X)(\partial_\rho X)(\partial^\mu \partial^\nu \partial^\rho X)t^7 \\ - \frac{17}{5040}(\partial_\mu X)(\partial_\nu X)(\partial^\mu \partial^\rho X)(\partial^\nu \partial_\rho X)t^7 \quad (48)$$

This exponentiated form also correctly reproduces any terms that can be reduced to products of $O(D^6)$ terms. As might be expected, most of the terms in W_4 are also recovered in the proper time expansion of the heat kernel [8], or by other means [14]. However, most terms in W_6 are new and were obtained with very little effort.

The fact that the field strength has been regarded as a small quantity might be regarded as a drawback in the present approach. This can be overcome by a resummation of field strength terms to produce a small derivative expansion where F is of order D^0 .

First of all define a new propagator D_F ,

$$D_F(t_r, t_s) = \sum_{n=0}^{\infty} D_n(t_r, t_s) F^n \quad (49)$$

where F is a matrix and

$$D_n(t_r, t_s) = \int dt_1 \dots dt_n D(t_r, t_1) \vec{D}(t_1, t_2) \dots \vec{D}(t_n, t_s) \quad (50)$$

We also define \vec{D}_F by differentiating with respect to t_r . An explicit expression for D_F and is given in appendix C.

The contribution to the heat kernel from ring diagrams involving only the field strength is

$$W_0 = \sum_{n=1}^{\infty} \frac{1}{2n} \text{tr}(F^n) \int_0^t \vec{D}_{n-1}(t', t') dt' \quad (51)$$

We can write this in terms of the new propagator,

$$W_0 = \frac{1}{2} \int_0^F d\omega \int_0^t dt' \vec{D}_\omega(t', t') \quad (52)$$

By equation (C5),

$$W_0 = -\frac{1}{2} \text{tr} \log \frac{\sinh Ft}{Ft} \quad (53)$$

This recovers Schwinger's result for the heat kernel in a background with constant field strength [1].

In other diagrams, replacing the propagator D by D_F resums all of the terms involving F . For example,

$$W_2 = \frac{1}{2} \int dt_1 dt_2 X_{,\mu} X_{,\nu} D_F(t_1, t_2)^{\mu\nu} - \frac{1}{2} \int dt_1 \text{tr} X_{,\mu\nu} D_F(t_1, t_1)^{\mu\nu}. \quad (54)$$

Again, after using equation (C5),

$$W_2 = \left(\frac{1}{4} X_{,\mu} X_{,\nu} - \frac{1}{2} X_{,\mu\nu} \right) \left(F^{-1} t^2 \coth Ft - F^{-2} t \right)^{\mu\nu} \quad (55)$$

This new result reduces to the previous result if $F = 0$.

A quite different situation exists when the derivatives are large, $\partial^2 X \gg X^2$. The large derivative expansion of the heat kernel is then given by the diagrams shown in figure 4. By summing these diagrams it is possible to order the derivative expansion in powers of X , thus

$$G(x, x, t) \sim G_1(x, x, t) + G_2(x, x, t) + \dots \quad (56)$$

where

$$G_1(x, x, t) = K_0(t) \int d\mu(k_1) (-1) e^{ik_1 \cdot x} f(ik_1 t) t \hat{X}(k_1) \quad (57)$$

$$G_2(x, x, t) = K_0(t) \int d\mu(k_1) d\mu(k_2) e^{i(k_1+k_2) \cdot x} f(ik_1 t, ik_2 t) t^2 \hat{X}(k_1) \hat{X}(k_2) \quad (58)$$

We have written $\hat{X}(k)$ for the Fourier transform of X . According to the diagrammatic rules, the single vertex diagrams give

$$f(ik_1 t) = \sum_p \frac{1}{2^p} \frac{1}{p!} \int \frac{dt_1}{t} D(t_1, t_1)^p (ik_1)^{2p} \quad (59)$$

Similarly, for the two-vertex diagrams

$$f(ik_1 t, ik_2 t) = \sum_{p,q,r} \frac{1}{2^{q+r}} \frac{1}{q!r!p!} \int \frac{dt_1}{t} \frac{dt_2}{t} D(t_1, t_1)^q D(t_1, t_2)^p D(t_2, t_2)^r (ik_1)^{2q} (ik_2)^{2r} (-k_1 \cdot k_2)^p \quad (60)$$

Using the explicit form of the propagator (33), we can perform the summation to express $f(a)$ and $f(a, b)$ in closed form,

$$f(a) = \int_0^1 dx e^{x(1-x)a^2} \quad (61)$$

$$f(a, b) = 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 e^{x_1(1-x_1)a^2 + x_2(1-x_2)b^2 + 2x_1 x_2 a \cdot b}. \quad (62)$$

The local expression for the heat kernel in the large derivative limit given by equations (57) and (58) can be used in any region in which the derivatives are large, even if they are not large everywhere.

For the integrated heat kernel, $k_1 = -k_2$ and we can use $f(a, -a) = f(a)$ to obtain

$$\int d\mu(x) G(x, x, t) = K_0(t) \int d\mu(k) f(ikt) t^2 \hat{X}(k) \hat{X}(-k) \quad (63)$$

This demonstrates agreement between the average of our heat kernel and the results for the integrated heat kernel given in reference [12].

V. CONCLUSIONS

We have seen that a diagrammatic approach gives the covariant derivative expansion of the heat kernel in both the large and small derivative limits. The method gives the coincident limits of the heat kernel or derivatives of the heat kernel and can be applied to any minimal second order operator.

The new diagrammatic expansion reproduces known results quickly and reliably, and we have also obtained some new terms in the derivative expansion in flat space with constant or vanishing background gauge field strengths. The derivation of more new results is presently underway.

One important area of application involves finding the heat kernel at finite temperatures to evaluate thermodynamic functions. The new method generalises very easily to this situation by following the ideas presented in [15]. If only spatial derivatives are present, then the only change is the replacement of $K_0(t)$ by a Jacobi theta function,

$$K_0(t) = (4\pi t)^{(1-d)/2} \theta_3(0, 4\pi i \beta^{-2} t) \quad (64)$$

where β is the inverse temperature.

APPENDIX A: VERTEX ELIMINATION

In this appendix we examine how some of the vertices in the diagrammatic expansion can be eliminated by modifying the propagator.

The full set of terms which generate the vertices are given by equation (29),

$$\begin{aligned} Z_g &= \frac{1}{4} h^{\mu\nu} \delta'_\mu \delta'_\nu, & Z_\omega &= \frac{1}{2} \Gamma^\mu \delta'_\mu, & Z_a &= -\frac{1}{2} i \delta^{\mu\nu} \{\delta'_\mu, A_\nu\}, \\ Z_{ag} &= -\frac{1}{2} i h^{\mu\nu} \{\delta'_\mu, A_\nu\}, & Z_{\omega a} &= -i \Gamma^\mu A_\mu, \\ Z_{aa} &= -\delta^{\mu\nu} A_\mu A_\nu, & Z_{aga} &= -h^{\mu\nu} A_\mu A_\nu. \end{aligned} \quad (A1)$$

Modified brackets $\langle \dots \rangle_m$ are defined by applying Wick's theorem with a modified propagator

$$\overleftrightarrow{D} = 2\delta(t_i - t_j) - 2t^{-1} \quad (A2)$$

connecting the A vertices and with a restricted set of vertices generated by Z_g , Z_ω and Z_a only. Contractions between the δ' factors reproduce the missing vertices. Expanding the exponential in equation (9) with the reduced set of vertices gives a typical term

$$\frac{1}{p!q!r!} \langle T Z_g^p Z_\omega^q Z_a^r \rangle_m = \sum_{j \dots m} c_{jklm} \langle T Z_{ag}^j Z_{\omega a}^k Z_{aa}^l Z_{aga}^m Z_g^{p'} Z_\omega^{q'} Z_a^{r'} \rangle \quad (A3)$$

where

$$p' = p - j - m, \quad (A4)$$

$$q' = q - k, \quad (A5)$$

$$r' = r - j - k - 2l - 2m \quad (A6)$$

and the c_{jklm} are combinatorial factors. These factors can be found by counting the number of ways in which to divide up the vertices and then counting the possible contractions. The result is equal to the coefficient of the identical term obtained by expanding the time ordered exponential with the full set of vertices and the unmodified propagator.

APPENDIX B: NORMAL COORDINATE EXPANSIONS

It is well known that in a system of normal coordinates the partial derivatives of the metric and the gauge fields can be replaced by covariant expressions involving curvatures [17]. We will give a brief review here to bring out some important features and to give the results in a form that can be used in the diagrammatic expansion of the heat kernel.

The gauge covariant derivative along a coordinate basis \mathbf{e}_μ will be denoted by D_μ . The curvature operator \mathcal{R} can be defined by the covariant derivative acting on a vector field \mathbf{e}_a ,

$$[D_\mu, D_\nu]\mathbf{e}_a = \mathcal{R}(\mathbf{e}_\mu, \mathbf{e}_\nu)\mathbf{e}_a \quad (\text{B1})$$

By allowing the vector \mathbf{e}_a to point in either the direction tangential to the manifold or the direction of the internal symmetry allows the connection coefficients to include both the tetrad connection and the gauge field A_μ . The corresponding components of the curvature operator are the Riemann tensor and the Field strength tensor.

We choose a point P and an orthonormal frame \mathbf{e}_a at P to set up the normal coordinates. Consider a family of geodesics passing through P with tangent vectors ξ . The normal coordinates x^μ of a point Q are defined by

$$x^\mu = \sigma \xi^\mu(\tau) \quad (\text{B2})$$

where σ is the distance along the geodesic from P to Q and $\xi(\tau)$ is the unit tangent vector at P . This construction implies that the coordinate vectors \mathbf{e}_μ satisfy a commutation relation

$$D_\xi(\sigma \mathbf{e}_\mu) - \sigma D_\mu \xi = \lambda \xi \quad (\text{B3})$$

for a constant λ . We can also arrange that $D_\xi \mathbf{e}_a = 0$ for the reference frame used to calculate the curvature components.

By repeatedly differentiating σA_μ and $\sigma \mathbf{e}_\mu$ at P we get two important identities

$$D_\xi^{n-1} A_\mu = \frac{n-1}{n} D_\xi^{n-2} F(\xi, \mathbf{e}_\mu) \quad (\text{B4})$$

$$D_\xi^n \mathbf{e}_\mu = \frac{n-1}{n+1} D_\xi^{n-2} R(\xi, \mathbf{e}_\mu) \xi \quad (\text{B5})$$

These can be used recursively to allow us to replace the connection components and their derivatives with covariant derivatives of the field strengths.

If the manifold is flat then equation (B4) has an explicit solution

$$D_\xi^{n-1} A_\mu = \frac{n-1}{n} F_{\nu_1 \mu; \nu_2 \dots \nu_{n-1}} \xi^{\nu_1} \dots \xi^{\nu_{n-1}} \quad (\text{B6})$$

where the ‘;’ denotes the gauge covariant derivative. In this case the choice of gauge is identical to the gauge introduced originally by Fock and developed by Schwinger [18,19].

In the curved case, we are interested in derivatives of the inverse metric and the connection Γ^μ , where

$$\Gamma^\mu = g^{\mu\nu} \omega_{,\nu} - h^{\mu\nu}_{,\nu} \quad (\text{B7})$$

and $\omega = \log |g^{\mu\nu}|^{1/2}$. We can use matrix notation, writing e for the tetrad and g for the metric, with $g = e^T e$ and $\omega = -\log |e|$. Covariant expressions for derivatives of the determinant and the inverse of the metric at P can then be obtained from Faa di Bruno's formula [20], for example

$$D^n \omega = \sum_{m=1}^n \sum_{\{a_i\}} \frac{(-1)^m}{m} \frac{n!}{(1!)^{a_1} \dots (n!)^{a_n}} \{ \text{tr}((De)^{a_1} \dots (D^n e)^{a_n}) + \text{permutations} \} \quad (\text{B8})$$

$$D^n g^{-1} = \sum_{m=1}^n \sum_{\{a_i\}} (-1)^m \frac{(m+1)n!}{(1!)^{a_1} \dots (n!)^{a_n}} \{ (De)^{a_1} \dots (D^n e)^{a_n} + \text{permutations} \} \quad (\text{B9})$$

summed over $a_1 + a_2 + \dots + a_n = m$ and $a_1 + 2a_2 + \dots + na_n = n$. Expressions for $D^n e$ can be obtained from equation (B5).

Our results are tabulated in tables I to III. These can be partially checked against similar expansions in a paper by van de Ven [9], which contains results up to $n = 6$. Our \mathbf{R}_j corresponds to $\frac{j+3}{j+1} \mathbf{K}_{j+2}$ and \mathbf{F}_j to $\frac{j+2}{j+1} \mathbf{Y}_{j+1}$ in his 'index-free' notation.

APPENDIX C: TIME INTEGRALS

Each diagram \mathcal{D} in the heat kernel expansion in derivatives corresponds to an expression of the form

$$G_{\mathcal{D}} = K_0(t) e^{-Xt} \int dt_1 \dots dt_n T \left(e^{Xt_1} Z_1 e^{-Xt_1} \dots e^{Xt_n} Z_n e^{-Xt_n} \right) g(t_1 \dots t_n) \quad (\text{C1})$$

We will consider the abelian case first.

When the terms commute we have

$$G_{\mathcal{D}} = K_0(t) e^{-Xt} Z_1 \dots Z_n \int dt_1 \dots dt_n g(t_1 \dots t_n) \quad (\text{C2})$$

where $g(t_1 \dots t_n)$ is a product of propagators. The integrals can be reduced recursively by equations such as

$$\int_0^t D(t_1, t_2)^p dt_2 = \frac{1}{p+1} D(t_1, t_1)^p \quad (\text{C3})$$

or similar results. We can also replace arrows over the propagators $D(t_i, t_j)$ by derivatives with respect to t_i for \rightarrow and t_j for \leftarrow . These simplifications reduce the integrals for terms up to $O(D^6)$ to those tabulated in table IV.

The results quoted in section 3 can be obtained from the fact that $D_\omega(t_i, t_j)$ satisfies

$$\frac{\partial^2 D_\omega}{\partial t_j^2} - 2\omega \frac{\partial D_\omega}{\partial t_j} = 2\delta(t_j - t_i) \quad (\text{C4})$$

and D_ω vanishes at 0 and t . Consequently

$$D_\omega(t_i, t_j) = \begin{cases} \frac{(e^{2\omega t_j} - 1)(e^{-2\omega t_i} - e^{-2\omega t})}{\omega(1 - e^{-2\omega t})} & t_i > t_j \\ \frac{(e^{2\omega(t_j-t)} - 1)(e^{-2\omega t_i} - 1)}{\omega(1 - e^{-2\omega t})} & t_i < t_j \end{cases} \quad (\text{C5})$$

Also $\overrightarrow{D}_\omega(t_i, t_j) = \partial_i D_\omega(t_i, t_j)$.

In the non-abelian case can need to introduce a sum over permutations π ,

$$G_{\mathcal{D}} = K_0(t) e^{-Xt} \sum_{\pi} F_{\pi}(L_i t) Z_{\pi(1)} \dots Z_{\pi(n)} \quad (\text{C6})$$

where L_i is the commutator $L_i Z_j = [X, Z_i] \delta_{ij}$ and

$$F_{\pi}(\alpha_i) = \int_{t_{\pi(1)} > \dots > t_{\pi(n)}} dt_1 \dots dt_n g(t_1 \dots t_n) e^{(\alpha_1 t_1 + \dots + \alpha_n t_n)/t} \quad (\text{C7})$$

The functions F_{π} depend on the diagram but not on the particular gauge group.

We can process the results further by using a gauge transformation to diagonalise the matrix X . It is then possible to choose a canonical basis E_a of the Lie algebra with the property that

$$[X, E_a] = w(a) E_a \quad (\text{C8})$$

for constants $w(a)$. The vertices which do not commute with X belong to representations of the Lie algebra of the gauge group, $Z_i = Z_i^a E_a$, therefore

$$G_{\mathcal{D}} = K_0(t) e^{-Xt} \sum_{a_1 \dots a_n} c_{a_1 \dots a_n} Z_1^{a_1} \dots Z_n^{a_n} \quad (\text{C9})$$

where

$$c_{a_1 \dots a_n} = \sum_{\pi} F_{\pi}(w(a_i) t) E_{a_{\pi(1)}} \dots E_{a_{\pi(n)}}. \quad (\text{C10})$$

We see from this expression that the range of values where F_{π} needs to be calculated is restricted by the Lie algebra roots.

TABLES

n	$D^n g^{-1}$
2	$-\frac{2}{3}R_0$
3	$-R_1$
4	$-\frac{6}{5}R_2 + \frac{8}{5}R_0^2$
5	$-\frac{4}{3}R_3 + 8R_0R_1$
6	$-\frac{10}{7}R_4 + \frac{100}{7}R_{(0}R_2) + \frac{85}{7}R_1^2 - \frac{160}{21}R_0^3$
7	$-\frac{3}{2}R_5 + \frac{122}{3}R_{(0}R_3) + \frac{357}{5}R_{(1}R_2) + \frac{6611}{45}R_{(0}^2R_1) + \frac{661}{9}R_0R_1R_0$
8	$-\frac{14}{9}R_6 + \frac{280}{9}R_{(0}R_4) + \frac{824}{9}R_{(1}R_3) + \frac{308}{5}R_2^2 - \frac{1596}{5}R_{(0}^2R_2)$ $-\frac{3052}{9}R_{(0}R_1^2) - \frac{868}{9}R_1R_0R_1 - \frac{4676}{9}R_0R_2R_0 + \frac{628}{5}R_0^4$

TABLE I. Expressions for $D_\xi^n g^{-1}$ in terms of curvatures. In this table, R_j denotes the linear mapping $R_j X = (D_\xi^j \mathcal{R})(\xi, X)\xi$, where \mathcal{R} is the curvature operator. In terms of components, $D^n g^{\mu\nu} = \mathbf{e}_\mu \cdot (D^n g^{-1})\mathbf{e}_\nu$ and $\mathbf{e}_\mu \cdot R_j \mathbf{e}_\nu = R_{\mu\nu_1\nu_2\nu;\nu_3\dots\nu_{j+2}}\xi^{\nu_1}\dots\xi^{\nu_{j+2}}$.

n	expression
2	$-\frac{1}{3}R_0$
3	$-\frac{1}{2}R_1$
4	$-\frac{3}{5}R_2 + \frac{2}{15}R_0^2$
5	$-\frac{2}{3}R_3 + \frac{2}{3}R_0R_1$
6	$-\frac{5}{7}R_4 + \frac{8}{7}R_0R_2 + \frac{15}{14}R_1^2 - \frac{16}{63}R_0^3$
7	$-\frac{3}{4}R_5 + 4R_0R_3 + \frac{9}{2}R_1R_2 - \frac{43}{3}R_0^2R_1$
8	$-\frac{7}{9}R_6 + \frac{20}{9}R_0R_4 + \frac{70}{9}R_1R_3 + \frac{28}{5}R_2^2 - \frac{272}{45}R_0^2R_2 - \frac{310}{9}R_0R_1^2 + \frac{16}{15}R_0^4$

TABLE II. Expressions for $D_\xi^n \omega$, where $\omega = \log |g^{\mu\nu}|^{1/2}$, are obtained by tracing the entries in this table.

n	$D^{n-1}A$
2	$\frac{1}{2}F_0$
3	$\frac{2}{3}F_1$
4	$\frac{3}{4}F_2 + \frac{1}{3}F_0R_0$
5	$\frac{4}{5}F_3 + \frac{4}{5}F_1R_0 + \frac{2}{5}F_0R_1$
6	$\frac{5}{6}F_4 + \frac{5}{3}F_2R_0 + \frac{5}{3}F_1R_1 + \frac{1}{2}F_0R_2 + \frac{1}{6}F_0R_0^2$
7	$\frac{6}{7}F_5 + \frac{20}{7}F_3R_0 + \frac{30}{7}F_2R_1 + \frac{18}{7}F_1R_2 + \frac{6}{7}F_1R_0^2 + \frac{4}{7}F_0R_3 + \frac{4}{21}F_0R_1R_0 + \frac{2}{7}F_0R_0R_1$
8	$\frac{7}{8}F_6 + \frac{35}{8}F_4R_0 + \frac{35}{4}F_3R_1 + \frac{63}{8}F_2R_2 + \frac{21}{8}F_2R_0^2 + \frac{7}{2}F_1R_3 + \frac{7}{2}F_1R_1R_0$ $+ \frac{7}{4}F_1R_0R_1 + \frac{5}{8}F_0R_4 + \frac{10}{8}F_0R_2R_0 + \frac{10}{8}F_0R_1^2 + \frac{3}{8}F_0R_0R_2 + \frac{1}{8}F_0R_0^3$

TABLE III. Expressions for $D_\xi^{n-1}A$ in terms of curvatures and field strengths. In this table F_j denotes the linear mapping $F_j X = (D_\xi^j \mathcal{R})(\xi, X)$ with gauge field strength indices. In terms of components, $D_\xi^n A_\mu = (D_\xi^n A)\mathbf{e}_\mu$ and $F_j \mathbf{e}_\mu = F_{\nu_1\mu;\nu_2\dots\nu_{j+1}}\xi^{\nu_1}\dots\xi^{\nu_{j+1}}$.

vertices	g	numerical factor	term
1	$D(t_1, t_1)^p$	$2^p(p!)^2/(2p+1)!$	t^{p+1}
	$D(t_1, t_1)^p \vec{D}(t_1, t_1)^{2q}$	$2^p p! (2q)! (p+q)! / (2p+2q+1)! q!$	t^{p+1}
2	$D(t_1, t_1) D(t_1, t_2) D(t_2, t_2)$	17/630	t^5
n	$D(t_1, t_2) \dots D(t_n, t_1)$	$(-2)^{3n-1} B_{2n} / (2n)!$	t^{2n}
$2n$	$\vec{D}(t_1, t_2) \dots \vec{D}(t_{2n}, t_1)$	$-2^{2n-1} B_{2n} / (2n)!$	t^{2n}

TABLE IV. The time integrals corresponding to diagrams for an abelian operator up to $O(D^6)$ can be reduced to the examples tabulated here. The results are given by the numerical factors multiplied by the terms in t , and B_n are Bernouilly numbers.

FIGURES

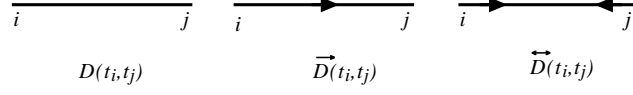


FIG. 1. Lines appearing in the expansion of the heat equation. They correspond to the two-point functions of time as indicated.

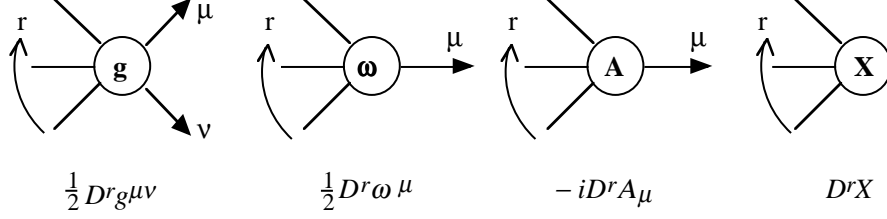


FIG. 2. Vertices appearing in the diagrammatic expansion. These vertices are attached to r plain lines and up to 2 lines with arrows. Each vertex corresponds to an integral over time and the vertex function indicated below the vertex. Covariant expressions can be found in the tables.

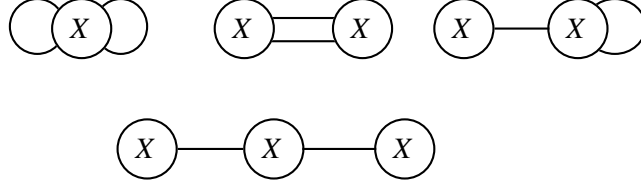


FIG. 3. The terms in the derivative expansion of $O(D^4)$ are formed from the diagrams shown here.

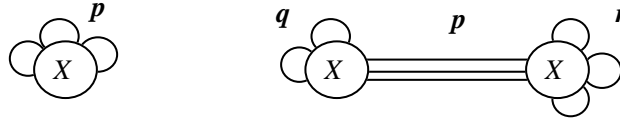


FIG. 4. Diagrams of this type dominate the large derivative approximation.

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